Towards Robust State Estimation in Matrix Lie Groups

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***Abstract*—In many practical scenarios, the sensors present in mobile robots encounter unknown disturbances due to faults, spoofing attacks, etc. which can lead to the accumulation of state estimation errors over time. In this work, we investigate the design of robust state estimation for systems whose dynamics can be embedded into matrix Lie groups. We consider time-varying uncertainty in both the state dynamics and the measurements. The exact log-linearization of the non-linear estimation error dynamics results in a non-linear evolution of the error in the presence of disturbance. We derive the conditions under which the non-linear evolution of estimation error is bounded and use Lyapunov stability theory to design a robust filter in the presence of unknown-but-bounded disturbance. We demonstrate the effectiveness of our approach through a simulation of rover dynamics embedded in the SE(2) Lie Group.**

***Index Terms*—Robust State Estimation, Matrix Lie Groups, Log-linearization, Lyapunov Stability.**

I. INTRODUCTION

The state estimation of non-linear systems has been a challenging problem for decades. The widely adopted method in the industry for non-linear state estimation is the extended Kalman filter (EKF). However, the EKF doesn't provide any theoretical guarantees for stability for cases where the dynamics evolve rapidly or where there exists a large deviation in the initial estimate of the state. For dynamical systems evolving on Lie groups, a special class of Kalman filter has been developed by Bonnabel et. al. [1]. These filters are known as invariant extended Kalman filters (IEKFs). Contrary to the EKF, the error dynamics of these filters are independent of the current estimate of the state. This property allows the IEKF to have improved estimation performance. The IEKF has been successfully applied to a variety of robotics applications such as pose estimation, underwater vehicle localization, visual-inertial navigation, etc. [2]–[4].

Subsequently, the recent works [5]–[7] show the convergence properties of the IEKF for deterministic systems. The authors employ the log-linear property of the invariant error dynamics to prove the stability and optimality of the filter. In [8], the authors have proposed invariant error propagation in the discrete-time form on special Euclidean groups, $SE(3)$ using the approximations to the *Baker-Campbell-Hausdorff* formula [9]. The authors in [4], [10] derived the closed-form expression of the evolution of the invariant error dynamics in the presence of disturbance for tracking control and estimation, respectively. The approach followed by the authors in [4] uses a heuristic to account for the non-linear distortion of noise or unknown disturbance entering through the input measurements. In this work, we analyze the properties of the distortion matrix for matrix Lie groups which have many applications in the field of robotics. We find the sufficient condition under which the distortion matrix is norm bounded. Our main contributions are as follows:

- We first derive the conditions under which the distortion matrix is norm bounded.
- We design an algorithm to estimate the maximum amplification caused by the distortion matrix for a deterministic system with unknown-but-bounded disturbance. We do this by calculating the maximum singular value of the distortion matrix using an LMI-based iterative process.

The rest of the paper is organized as follows. In Section II, we present the propagation model, log-linearization, and measurement model for the dynamics embedded in the matrix Lie groups. Section III presents the design of a robust state estimator in matrix Lie groups and describes the algorithm for finding the maximum singular value of the distortion matrix. We show the effectiveness of the proposed method through simulations on rover dynamics embedded in the $SE(2)$ Lie group in Section IV. Finally, Section V summarizes the paper and proposes future directions.

Notations and Preliminaries: A matrix Lie Group denoted by $\mathcal{G}$ is a subset of $N \times N$ invertible square matrices [11]. It has the following properties:

$$\begin{aligned} I_N \in \mathcal{G}, \qquad \forall \chi \in \mathcal{G}, \chi^{-1} \in \mathcal{G}, \\ \forall \chi_1, \chi_2 \in \mathcal{G}, \quad \chi_1 \chi_2 \in \mathcal{G}, \end{aligned} \tag{1}$$

where I_N denotes the identity matrix of $\mathbb{R}^N$. For every element $\chi \in \mathcal{G}$, there is an associated vector space $T_\chi \mathcal{G}$ that is called the tangent space at χ. The tangent space at the identity element I_N is called the Lie algebra denoted by $\mathfrak{g}$. Its dimension is denoted as d. There is a linear bijection between Euclidean space $\mathbb{R}^d$ to $\mathfrak{g}$ denoted as $[.]^\wedge$, such that for $\zeta \in \mathbb{R}^d$ and $[\zeta]^\wedge \in \mathfrak{g}$ there is a linear map $\zeta \longmapsto [\zeta]^\wedge$. $\|.\|$ denotes the 2-norm and induced 2-norm for vectors and matrices, respectively. $\|.\|_\infty$ denotes the $\mathcal{L}^\infty$ norm.

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II. Problem Setup

A. Propagation Model

We consider a mixed-invariant system [8] on $\mathcal{G}$ defined as:

$$\dot{\chi}_t = \chi_t[u_t^l]^\wedge + [u_t^r]^\wedge \chi_t \tag{2}$$

where $\chi_t \in \mathcal{G}$ are the states of the system and $[u_t^r]^\wedge, [u_t^l]^\wedge \in \mathfrak{g}$ are the inputs to the system at time t.

Remark. Note that the mixed invariant dynamics (2) are well suited for modeling the dynamics of mechanical systems in which some inputs are measured in a body-fixed frame, i.e., u_t^l while others are measured in the inertial frame, i.e., u_t^r. [8]

Consider two distinct trajectories of (2), χ_t and $\hat{\chi}_t$. We define two errors η_t^l and η_t^r as the left and right invariant errors, similar to [6], as:

$$\eta_t^l = \chi_t^{-1}\hat{\chi}_t \tag{3}$$

$$\eta_t^r = \hat{\chi}_t\chi_t^{-1} \tag{4}$$

The idea behind the terminology of invariant errors comes from the invariance of (3) to left multiplication $(\chi, \bar{\chi}) \to (\Gamma\chi, \Gamma\bar{\chi})$ and (4) to right multiplication $(\chi, \bar{\chi}) \to (\chi\Gamma, \bar{\chi}\Gamma)$, respectively, where $\Gamma \in \mathcal{G}$. The left-invariant and right-invariant errors have a state-trajectory independent propagation if they satisfy $\dot{\eta}_t = f(\eta_t)$ where $f(.)$ denotes a function of η_t. If $\hat{\chi}_t$ denotes estimated states, then the left (right) invariant error η_t^l (η_t^r) represents the left (right) state estimation error, respectively.

$$\dot{\hat{\chi}}_t = \hat{\chi}_t[u_t^l + \mathbf{w}^l]^\wedge + [u_t^r + \mathbf{w}^r]^\wedge\hat{\chi}_t \tag{5}$$

where $\mathbf{w}^l, \mathbf{w}^r \in \mathbb{R}^d$ represent the noise (or unknown disturbance) in the input measurements.

Definition 1 (Adjoint Operator)**.** For any $\chi \in \mathcal{G}$ and $\zeta \in \mathbb{R}^d$, there exists a linear map $Ad_\chi : \mathfrak{g} \to \mathfrak{g}$ defined as $Ad_\chi([\zeta]^\wedge) = \chi^{-1}\zeta^\wedge\chi$. The adjoint operation is often represented in a matrix form as $Ad_\chi([\zeta]^\wedge) = [Ad_\chi\zeta]^\wedge$

Definition 2 (Group-Affine System)**.** The system with the dynamics $\dot{\chi} = f(\chi)$ is said to be group affine if it satisfies $f(AB) = f(A)B + Af(B) + Af(I)B$ where $A, B, X \in \mathcal{G}$, and I is the identity element of the Lie Group. For group-affine systems, the left and right invariant errors are independent of the current state [6].

The mixed invariant dynamics defined in (2) satisfies the group affine property as follows. Let $f(\chi_t) = \chi_t[u_t^l]^\wedge + [u_t^r]^\wedge\chi_t$. Then, we have

$$\begin{aligned} f(A)B + Af(B) - Af(I)B = (A[u_t^l]^\wedge + [u_t^r]^\wedge A)B \\ +A(B[u_t^l]^\wedge + [u_t^r]^\wedge B) - A([u_t^l]^\wedge + [u_t^r]^\wedge)B = f(AB) \end{aligned}$$

where $A, B, X, I \in \mathcal{G}$. For the dynamical system defined in (2), it can be shown that the dynamics of the invariant errors can be written as:

$$\dot{\eta}_t^l = -[u_t^l]^\wedge\eta_t^l + \eta_t^l[u_t^l + \mathbf{w}^l]^\wedge - Ad_{\chi^{-1}}[\mathbf{w}^r]^\wedge\eta_t^l \quad \text{(Left)} \tag{6}$$

$$\dot{\eta}_t^r = -\eta_t^r[u_t^r]^\wedge + [u_t^r + \mathbf{w}^r]^\wedge\eta_t^r + \eta_t^r Ad_\chi[\mathbf{w}^l]^\wedge \quad \text{(Right)} \tag{7}$$

Definition 3 (Exponential and Logarithm Map)**.** The exponential map of a matrix, exp: $\mathfrak{g} \to \mathcal{G}$ is a bijection in the neighbourhood of $0 \in \mathbb{R}^d$ to the neighbourhood of $I_N \in \mathcal{G}$. Similarily, the logarithm map is defined as log: $\mathcal{G} \to \mathfrak{g}$. It is the inverse of the exponential map in the same neighbourhood.

For an element $\eta_t \in \mathcal{G}$, the exponential map provides a bijection between a neighbourhood of $\mathbb{R}^d$ and a neighbourhood of $I \in \mathcal{G}$. The estimation error can be approximated by an element $\zeta_t \in \mathbb{R}^d$ such that $\eta_t = \exp([\zeta_t]^\wedge)$.

B. Log Linearization of the error dynamics

For the sake of brevity, we consider only the log-linearization of the left-invariant error dynamics. The steps involved for the right-invariant error dynamics follow the same procedure.

Lemma 1 (Log-linearization of Left-Invariant Error Dynamics [4], [10])**.** *For the dynamical system defined in (2) and invariant error dynamics defined in (6), the error dynamics of the $[\zeta_t^l]^\wedge \in \mathfrak{g}$ is defined as:*

$$[\dot{\zeta}_t^l]^\wedge = \mathbf{A}[\zeta_t^l]^\wedge - \mathbf{U}_\zeta[\mathbf{w}^l + Ad_{\chi_t^{-1}}\mathbf{w}^r]^\wedge \tag{8}$$

where $\mathbf{A} = -ad_{[u_t^l]^\wedge}$ and $\mathbf{U}_\zeta := \left(\sum_{i=0}^{\infty} \frac{(-1)^i}{(i+1)!}(ad_{[\zeta_t^l]^\wedge})^i\right)^{-1}$. We call $\mathbf{U}_\zeta$ the distortion matrix.

Proof. The proof utilizes the results of the derivative of the exponential map [12]. It is omitted and can be found in [4], [10]. □

Remark. It can be observed that when $\zeta_t \approx 0$, the matrix becomes identity $\mathbf{U}_\zeta \approx \mathbf{I}$, and the results obtained in (8) becomes equivalent to the results in [6].

We now present the conditions under which the matrix $\mathbf{U}_\zeta$ is norm-bounded.

Lemma 2. *Consider $\zeta \in \mathfrak{g}$ to be an element in the Lie algebra. Let λ_i denote the eigenvalues of ad_ζ. Then, the eigenvalues of $\mathbf{U}_\zeta^{-1}$ are given by $\frac{1-e^{-\lambda_i}}{\lambda_i}$*

Proof. Let $\mathbf{x}_i \in \mathbb{R}^d$ denote the eigenvector corresponding to the eigenvalue λ_i. Then, the following equation holds,

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{(i+1)!}(ad_{[\zeta_t^l]^\wedge})^i\mathbf{x}_i = \sum_{i=0}^{\infty} \frac{(-1)^i}{(i+1)!}(\lambda)^i\mathbf{x}_i \tag{9}$$

$$= \frac{1-e^{-\lambda_i}}{\lambda_i}\mathbf{x}_i \tag{10}$$

Also, the eigenvalues of the $\mathbf{U}_\zeta$ matrix are equal to $\frac{\lambda_i}{1-e^{-\lambda_i}}$, when $\lambda_i \neq 2k\pi i, k = \pm1, \pm2, \ldots$. □

Theorem 1. *For any $\zeta \in \mathfrak{g}$, if the eigenvalues of ad_ζ denoted by λ_i satisfy*

$$\lambda_i \neq 2k\pi i, k = 0, \pm1, \pm2, \ldots$$

The matrix norm $\|\mathbf{U}_\zeta\|$ is upper bounded by $\frac{|\lambda_{max}(ad_\zeta)|}{2}$ where $|\lambda_{max}(ad_\zeta)|$ denotes the magnitude of the maximum eigenvalue of ad_ζ.

Proof. The proof is given in Appendix A. □

Remark. For $SE(2), SE(3)$ and $SO(3)$ Lie groups, the eigenvalues of the matrix ad_ζ exist either in pairs of pure imaginary values or zero. The magnitude of the maximum eigenvalue of the matrix ad_ζ for various cases is shown in Table I.

TABLE I: Values of magnitude of maximum eigenvalue of ad_ζ

Lie algebra	Representation	Max. Eigenvalue $\lambda_{max}(ad_\zeta)$
$\mathfrak{se}(2)$	$\zeta := (u, \omega)$	$\lvert\omega\rvert$
$\mathfrak{so}(3)$	$\zeta := (\omega_\psi, \omega_\theta, \omega_\phi)$	$(\omega_\psi^2 + \omega_\theta^2 + \omega_\phi^2)^{\frac{1}{2}}$
$\mathfrak{se}(3)$	$\zeta := (u, \omega_\psi, \omega_\theta, \omega_\phi)$	$(\omega_\psi^2 + \omega_\theta^2 + \omega_\phi^2)^{\frac{1}{2}}$

As all the angular measurements are computed in the range of $[-\pi, \pi]$, the maximum eigenvalue of the matrix $|\lambda_{max}(ad_\zeta)| < \lambda_0$ where λ_0 is a constant. Using Lemma 2, we obtain that the matrix norm $\|U_\zeta\|$ is bounded for $\zeta \in \mathfrak{se}(2), \mathfrak{se}(3)$ and $\mathfrak{so}(3)$.

C. Measurement Model

We consider the same class of measurement model as considered in [6]. There are two different types of observation models, i.e., left and right invariant observations.

$$Y_t = \chi_t(b + \mathbf{v}) \quad \text{(Left Invariant)} \tag{11}$$

$$Y_t = \chi_t^{-1}(b + \mathbf{v}) \quad \text{(Right Invariant)} \tag{12}$$

where $b \in \mathbb{R}^d$ is a known vector and $\mathbf{v}$ denotes the noise (or unknown disturbance) present in the measurements. Considering the left-invariant observations, the innovation vector z_t^l can be defined as,

$$\begin{aligned} z_t^l &= \hat{\chi}_t^{-1} Y_t - \hat{\chi}_t^{-1} \hat{\chi}_t b \\ &= \eta_t^{-1}(b + \mathbf{v}) - b \end{aligned} \tag{13}$$

Using the first-order approximation, we get the following linear relationship for the innovation vector,

$$z_t^l \cong \mathbf{C}\zeta_t^l \ + \mathbf{v} \tag{14}$$

where $\mathbf{C}\zeta_t^l := [\zeta_t]^\wedge b$ is a linear operation on ζ_t^l.

III. Robust State Estimation in Matrix Lie Group

We start with the assumption that unknown disturbances in the measurements are bounded and derived from a deterministic process. Let $\bar{\mathbf{w}}$ and $\bar{\mathbf{v}}$ denote the deterministic unknown bounded disturbance. Then, we have, $\|\bar{\mathbf{w}}\| \leq \|\bar{\mathbf{w}}\|_\infty$ and $\|\bar{\mathbf{v}}\| \leq \|\bar{\mathbf{v}}\|_\infty$. Further, we restrict our analysis to the class of dynamical systems $\zeta \in \mathfrak{g}$ for which $|\lambda_{max}(ad_\zeta)| < \lambda_0$, where λ_0 is a constant, holds. We rewrite the dynamics of estimation error propagation in (8) and the innovation equation in (14) for the deterministic case considering only the left-invariant disturbance on error. To simplify notation, let $\zeta := [\zeta_t^l]^\wedge, u := [u_t^l]^\wedge$ and $z := z_t^l$

$$\begin{aligned} \dot{\zeta} &= \mathbf{A}\zeta + \mathbf{U}_\zeta \bar{\mathbf{w}} \\ z &= \mathbf{C}\zeta + \bar{\mathbf{v}} \end{aligned} \tag{15}$$

A. State Estimation Under Linearized Dynamics

The linear Kalman filter theory can be applied to the error dynamics and the innovation defined in (15). In the standard Kalman filter theory, the filters are designed for the stochastic systems with noisy measurements. Here, we consider a deterministic system being perturbed with unknown-but-bounded disturbances. In the deterministic case, the noise covariance matrices defined by $\mathbf{Q}$ and $\mathbf{R}$ becomes the gain tuning parameters which users are free to choose. The matrix $\mathbf{P}$ which denotes the error covariance in the standard Kalman filter, does not have a rigorous interpretation in the deterministic case. However, in practice it still conveys the information regarding the error propagation which may be useful. The procedure to compute the Kalman gain follows,

Prediction Step:

$$\begin{aligned} \dot{\zeta} &= \mathbf{A}\zeta + \mathbf{U}_\zeta \bar{\mathbf{w}} \\ \dot{\mathbf{P}} &= \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{U}_\zeta^T \mathbf{Q} \mathbf{U}_\zeta \end{aligned} \tag{16}$$

Update Step:

$$\begin{aligned} \dot{\zeta}^+ &= \mathbf{A}\zeta + \bar{\mathbf{w}} + \mathbf{L}(\mathbf{C}\zeta + \bar{\mathbf{v}}) \\ \dot{\mathbf{P}}^+ &= (\mathbf{I} - \mathbf{L}\mathbf{C})\mathbf{P} \\ \mathbf{S} &= \mathbf{C}\mathbf{P}\mathbf{C}^T + \mathbf{R} \\ \mathbf{L} &= \mathbf{P}\mathbf{C}^\mathbf{T}\mathbf{S}^{-\mathbf{1}} \end{aligned} \tag{17}$$

The estimated state for the original system in the Lie group $\hat{\chi}_t^l \in \mathcal{G}$ corresponding to the log linearized error dynamics (16), (17) after the measurement update has the following form:

$$(\hat{\chi}_t^l)^+ = \hat{\chi}_t^l \exp(\mathbf{L} z_t^l) \tag{18}$$

The term $\mathbf{U}_\zeta^T \mathbf{Q} \mathbf{U}_\zeta$ in the propagation step (16) depends upon the error dynamics ζ and modifies the matrix $\mathbf{Q}$. In this work, we over-approximate $\mathbf{U}_\zeta^T \mathbf{Q} \mathbf{U}_\zeta$ with its maximum singular value such that $\mathbf{U}_\zeta^T \mathbf{Q} \mathbf{U}_\zeta \leq \sigma_{max}^2(\mathbf{U}_\zeta)\mathbf{Q}$. We describe the procedure to calculate the maximum singular value in the next subsection.

Remark. In linear Kalman filter theory, inflating the covariance results in enhancing the robustness of the filter in the presence of unknown disturbances. However, it might lead to the degradation of performance of the filter (sometimes even divergence of the filter) if the inflation is not carefully designed.

B. LMIs under bounded disturbance

We use the Lyapunov stability analysis to find the maximum singular value of the $\mathbf{U}_\zeta$ matrix using an iterative process described in Algorithm 1 similar to [4]. Let the Lyapunov function for the linear system in (17) be defined as:

$$V(\zeta) = \zeta^T \mathbf{P} \zeta \tag{19}$$

where P is a real positive definite matrix. The derivative of the Lyapunov function can be written as:

$$\begin{aligned} \dot{V} &= \dot{\zeta}^T \mathbf{P} \zeta + \zeta^T \mathbf{P} \dot{\zeta} \\ &= (\zeta^T \bar{\mathbf{A}}^T + \bar{\mathbf{w}}^T \mathbf{U}_\zeta^T + \bar{\mathbf{v}}^T \mathbf{L}^T)\mathbf{P}\zeta + \zeta^T \mathbf{P}(\bar{\mathbf{A}}\zeta + \mathbf{U}_\zeta \bar{\mathbf{w}} + \mathbf{L}\bar{\mathbf{v}}) \\ &= \zeta^T(\bar{\mathbf{A}}^T\mathbf{P} + \mathbf{P}\bar{\mathbf{A}})\zeta + 2\zeta^T \mathbf{P}\mathbf{U}_\zeta \bar{\mathbf{w}} + 2\zeta^T \mathbf{P}\mathbf{L}\bar{\mathbf{v}} \end{aligned} \tag{20}$$

where $\bar{\mathbf{A}} \coloneqq \mathbf{A} + \mathbf{LC}$

Lemma 3. *Consider the system defined in (17). If there exist a continuously differentiable function $V(\zeta)$, and the real numbers $\alpha, \mu > 0$, and $\sigma_{max} \coloneqq \sigma_{max}(\mathbf{U}_\zeta)$ such that*

$$V(\zeta) \geq \mu_1 \sigma_{max}^2 \|\bar{\mathbf{w}}\|_\infty^2 + \mu_2 \|\bar{\mathbf{v}}\|_\infty^2 \tag{21}$$

and

$$\dot{V}(\zeta, t) \leq -\alpha V(\zeta) \tag{22}$$

are satisfied, then, for every bounded disturbances $\bar{\mathbf{w}}$ and $\bar{\mathbf{v}}$, the state ζ is bounded and the set

$$\{\zeta \in \mathbb{R}^d : V(\zeta) \leq \mu_1 \sigma_{max}^2 \|\bar{\mathbf{w}}\|_\infty^2 + \mu_2 \|\bar{\mathbf{v}}\|_\infty^2\} \tag{23}$$

is invariant and attractive for the system.

Proof. The proof is given in Appendix B. □

Using Lemma 3 and S-procedure [13], we can construct the following LMI combining (21) and (22) together as,

$$\begin{bmatrix} \bar{\mathbf{A}}^T\mathbf{P} + \mathbf{P}\bar{\mathbf{A}} + \alpha\mathbf{P} & \mathbf{P} & \mathbf{PL} \\ \mathbf{P} & -\alpha\mu_1\sigma_{max}^2\mathbf{I} & \mathbf{0} \\ \mathbf{L^T P} & \mathbf{0} & -\alpha\mu_2\mathbf{I} \end{bmatrix} \leq 0 \tag{24}$$

We start with an initial guess of the maximum singular value of $\mathbf{U}_\zeta$. We then iteratively solve the LMI till the value of $\sigma_{max}(\mathbf{U}_\zeta)$ converges.

Algorithm 1 Maximum Singular Value of $\mathbf{U}_\zeta$

Require: System matrices $\mathbf{A}$ and $\mathbf{C}$, Kalman gain $\mathbf{L}$, constants μ_1, μ_2, initial guess σ_0 and tolerance ϵ

```
σ_prev ← 0
σ_curr ← σ0                                   ▷ Set current as initial guess
while |σ_curr − σ_prev| ≥ ϵ do               ▷ Iterate till convergence
    Calculate the reachable set of ζ using σ_curr by solving
    the LMI given in (24)
    σ_prev ← σ_curr                           ▷ Set previous as current
    Find the maximum singular value σ_max(U_ζ) by
    searching through all the elements in the reachable set.
    σ_curr ← σ_max                            ▷ Set current as maximum
end while
```

IV. Simulation

Consider a rover model evolving on the 2D plane given as:

$$\frac{d}{dt}p_x = u_x \cos\theta - u_y \sin\theta,\ \frac{d}{dt}p_y = u_x \sin\theta + u_y \cos\theta,\ \frac{d\theta}{dt} = \omega$$

where p_x, p_y denote the positions and θ denotes the heading angle of the rover. u_x, u_y, ω represent the linear and angular velocity, respectively. The system can be embedded in the 2D special Euclidean group, $SE(2)$. The dynamics can be written in a left-invariant system form $\dot{\chi} = \chi[u]^\wedge$ where χ represents the state of rover in the $SE(2)$ Lie group and $u = [u_x, u_y, \omega]^T$ represents the rover's input, respectively. The measurements include the linear and angular velocity and the position measurements, respectively. We consider an unknown bounded disturbance on the input measurements such that the estimated state has the form $\dot{\hat{\chi}} = \hat{\chi}[u + \bar{\mathbf{w}}]^\wedge$ with $\bar{\mathbf{w}} = [\bar{w}_x, \bar{w}_y, \bar{w}_\omega]^T$ representing the unknown disturbance in the linear and angular velocities, respectively. The position measurement has the form described by $Y = \chi b$ where $\chi \in \mathcal{G}$ and $b = [0, 0, 1]^T$ is a known vector. We do not consider any disturbance in the position measurement for the sake of simplicity. For the simulations, we consider that the rover drives along a circular trajectory of diameter $10m$ for 40 seconds. To achieve this, a constant velocity control of $u = [0.7, 0, 0.14]$ is applied. The log-linearized error dynamics and the innovations in the Lie algebra for the state estimator of the rover can be described as:

$$\dot{\zeta} = A\zeta + \bar{\mathbf{w}}, \quad z = C\zeta$$

$$A = -\begin{bmatrix} 0 & -\omega & u_y \\ \omega & 0 & -u_x \\ 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} \mathbf{I}_2 & 0_{2,1} \end{bmatrix}$$

We consider the maximum disturbance in the measured input in two parts $\|\bar{\mathbf{w}}_u\|_\infty$ and $\|\bar{\mathbf{w}}_\omega\|_\infty$, one affecting the linear velocity input u_x, u_y and other affecting the angular velocity input ω. For the unknown disturbance, we use sinusoidal disturbances of varying frequencies. We simulate two different cases, i.e., a small disturbance $\|\bar{\mathbf{w}}_u\|_\infty = 0.01$ and a large disturbance $\|\bar{\mathbf{w}}_u\|_\infty = 0.05$. We use the filter defined in (17) to compute the gain matrix. In Figure 1, we can see that the estimated states are bounded under bounded disturbance even after being influenced by the distortion matrix.

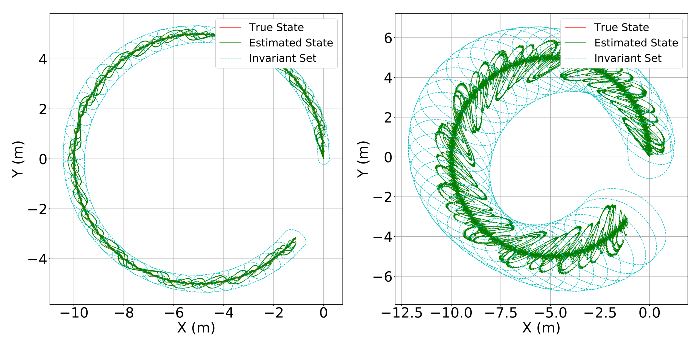

Fig. 1: Estimated states under bounded input disturbance. **Left**: Small Disturbance **Right**: Large Disturbance

V. Conclusion

In this paper, we showed that the log-linearized dynamics of estimation error in matrix Lie groups evolves non-linearly in the presence of unknown disturbances. We then derived the conditions under which the non-linear term in the log-linearized estimation error dynamics is bounded. We also described the procedure for finding the maximum amplification caused by the non-linear term by computing the maximum singular value of the distortion matrix using an iterative process. In the future, we would like to derive the conditions for stochastic stability of the estimation error dynamics in the presence of noise or unknown input with known stochastic properties.

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APPENDIX

A. Proof of Theorem 1

Consider $\mathbf{U}_\zeta$ as the distortion matrix defined in Lemma 1. Let $\sigma_{min}(\mathbf{U}_\zeta)$ and $\sigma_{max}(\mathbf{U}_\zeta)$ denote the minimum and the maximum singular values of $\mathbf{U}_\zeta$. Similarly, $\lambda_{min}(\mathbf{U}_\zeta)$ and $\lambda_{max}(\mathbf{U}_\zeta)$ denote the minimum and the maximum eigenvalues of $\mathbf{U}_\zeta$. We use the fact that the matrix norm is upper bounded by the maximum singular value of the matrix.

$$\|\mathbf{U}_\zeta\| \leq \sigma_{max}(\mathbf{U}_\zeta) \tag{25}$$

For a square matrix, the maximum singular value is equal to the minimum singular value of the inverse of the matrix. Consider $\mathbf{A} = \mathbf{U}^T\mathbf{\Sigma}\mathbf{V}$, then $\mathbf{A}^{-1} = \mathbf{V}^{-1}\mathbf{\Sigma}^{-1}\mathbf{U}^{-T} = \mathbf{V}^T\mathbf{\Sigma}^{-1}\mathbf{U}$ as the matrix $\mathbf{U}$ and $\mathbf{V}$ are unitary matrices. This implies that,

$$\|\mathbf{U}_\zeta\| \leq \frac{1}{\sigma_{min}(\mathbf{U}_\zeta^{-1})} \tag{26}$$

Using *Fan-Hoffman Inequality* (see Theorem 2 in [14]) for real matrices, we have

$$\|\mathbf{U}_\zeta\| \leq \frac{1}{\sigma_{min}(\mathbf{U}_\zeta^{-1})} \leq \frac{1}{|\lambda_{min}(\mathbf{U}_\zeta^{-1})|} \tag{27}$$

It can be observed from lemma 2 that $|\lambda_{min}(\mathbf{U}_\zeta^{-1})| = \frac{1-e^{-|\lambda_{max}(ad_\zeta)|}}{|\lambda_{max}(ad_\zeta)|}$. This implies that

$$\|\mathbf{U}_\zeta\| \leq \frac{|\lambda_{max}(ad_\zeta)|}{1-e^{-|\lambda_{max}(ad_\zeta)|}} \leq \frac{|\lambda_{max}(ad_\zeta)|}{2} \tag{28}$$

which completes the proof.

B. Proof of Lemma 3

Consider the system defined in (17) of $\zeta \in \mathfrak{g}$, We know that the matrix defined by $\|\mathbf{U}_\zeta\|$ is bounded such that,

$$\|\mathbf{U}_\zeta\| \leq \sigma_{max}(\mathbf{U}_\zeta) \tag{29}$$

We have $\|\mathbf{U}_\zeta\bar{\mathbf{w}}\|^2 \leq \sigma_{max}^2\|\bar{\mathbf{w}}\|_\infty$ and $\|\bar{\mathbf{v}}\|^2 \leq \|\bar{\mathbf{v}}\|_\infty^2$ as the maximum norms of the disturbances.

Using *LaSalle's Invariance Principle* [15], we get the largest invariant set in (23) for,

$$V(\zeta) > \mu_1\sigma_{max}^2\|\bar{\mathbf{w}}\|_\infty^2 + \mu_2\|\bar{\mathbf{v}}\|_\infty^2 \tag{30}$$

and

$$\dot{V}(\zeta, t) \leq -\alpha V(\zeta) \tag{31}$$

which completes the proof.

C. Maximum Singular Value of $\mathbf{U}_\zeta$ for $SE(2)$

For $\zeta := (\zeta_x, \zeta_y, \zeta_\theta) \in \mathfrak{se}(2)$, the closed-form expression for U_ζ for $SE(2)$ Lie group can be written as:

$$U_\zeta = \begin{bmatrix} a & -b & \frac{\zeta_\theta\zeta_x\sin(\zeta_\theta)+(1-\cos(\zeta_\theta))(\zeta_\theta\zeta_y-2\zeta_x)}{2\zeta_\theta(1-\cos(\zeta_\theta))} \\ b & a & \frac{\zeta_\theta\zeta_y\sin(\zeta_\theta)+(1-\cos(\zeta_\theta))(-\zeta_\theta\zeta_x-2\zeta_y)}{2\zeta_\theta(1-\cos(\zeta_\theta))} \\ 0 & 0 & -1 \end{bmatrix} \tag{32}$$

where $a = \frac{\zeta_\theta\sin(\zeta_\theta)}{2(\cos(\zeta_\theta)-1)}$ and $b = \frac{\zeta_\theta}{2}$.

The maximum eigen value of the matrix U_ζ has the form defined as,

$$\sigma_{max}(\mathbf{U}_\zeta) = \frac{\zeta_\theta}{2(1-cos(\zeta_\theta))} \tag{33}$$